Exercises for 'Functional Analysis 2' [MATH-404]

(24/02/2024)

Ex 2.1 (Topological vector spaces induced by seminorms)
Let X be a vector space equipped with a family of seminorms $(p_i)_{i\in I}$. Define a topology τ on X by setting

$$U \in \tau \iff \forall\, x \in U\ \exists I_0 \subset I \text{ finite },\ \varepsilon > 0 :\ B_{\varepsilon,I_0}(x) \subset U$$

with $B_{\varepsilon,I_0}(x) = \{y \in X : p_i(x-y) < \varepsilon \quad \forall i \in I_0\}$. Show that this notion indeed defines a topology on X and that (X,τ) becomes a topological vector space.

Ex 2.2 (The weak topology on a Banach space as LCTVS)
Let $(X, \|\cdot\|)$ be a Banach space (over $\mathbb{R}$). Recall that the **weak topology** on X is the coarsest topology such that all linear functionals $f : X \to \mathbb{R}$ that are continuous with respect to the norm convergence remain continuous. Show that X equipped with the weak topology becomes a locally convex topological vector space.

Hint: Construct seminorms inducing the weak topology. A corollary of the Hahn–Banach Theorem might be useful to separate points.

Ex 2.3 (L^p spaces for $0 < p < 1$)
Let $(\Omega, \mathscr{A}, \mu)$ be a measure space and let $p \in (0,1)$. Define

$$L^p(\mu) = \Big\{f : \Omega \to \mathbb{R} : \int_\Omega |f|^p\, d\mu < +\infty\Big\},$$
$$\rho(f) = \int_\Omega |f|^p\, d\mu.$$

As usual, we identify the functions that are equal μ-almost everywhere.

a) Prove that $L^p(\mu)$ is a vector space and that $d(f,g) = \rho(f-g)$ is a translation-invariant metric on $L^p(\mu)$.
Hint: For $p \in (0,1)$ the estimate $(s+t)^p \le s^p + t^p$ holds for all $s,t \ge 0$

b) Show that the topology induced by d turns $L^p(\mu)$ into a TVS.

c) Assume that μ is the Lebesgue measure on $\Omega = \mathbb{R}$. Show that for every $\delta > 0$

$$\sup\big\{\rho(f) :\ f \in \mathrm{co}(B_\delta)\big\} = +\infty,$$

where $B_\delta = \{f :\ \rho(f) < \delta\}$ and $\mathrm{co}(B_\delta)$ is the convex hull of B_δ.
Hint: Consider for some $\lambda > 0$ the functions $g_n = \lambda\chi_{[n,n+1)}$ and certain convex combinations.

Ex 2.4 (LCTVS with countable family of seminorms is metrizable)
Let X be a LCTVS with the topology defined by a countable family of seminorms $(p_n)_{n\in\mathbb{N}}$.

a) Consider the function $f(a) = a/(1+a)$, $a \geq 0$. Show that
$$f(a) \leq f(a+b) \leq f(a) + f(b).$$
for all $b \geq 0$.

b) Show that
$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x-y)}{1+p_n(x-y)}$$
is a translation-invariant metric on X and the balls in this metric are balanced.
Hint: To demonstrate various properties of d it is convenient to prove instead the respective properties of the function $d_0(x) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x)}{1+p_n(x)}$, and use the identity $d(x,y) = d_0(x-y)$.

c) Verify that the metric topology induced by d is the same as the topology defined by the seminorms $(p_n)_{n\geq 1}$.

d) Show that
$$d_1(x,y) = \sum_{n=1}^{\infty} \min\left\{2^{-n}, p_n(x-y)\right\}$$
is likewise a translation-invariant metric defining the same topology.

Ex 2.5 (Two counterexamples)

a) **A metric-vector space but not TVS**
Consider the plane $\mathbb{R}^2$ with the "Washington" metric
$$d(x,y) = \begin{cases} \|x-y\| & \text{if } x \text{ and } y \text{ are colinear,} \\ \|x\| + \|y\| & \text{otherwise.} \end{cases}$$
Show that scalar multiplication is continuous, but addition is not even separately continuous in this metric.

b) **Balls in metrizable LCTVS may be non-convex**
Consider $C(\mathbb{R})$ with a countable family of seminorms
$$p_n(f) = \sup\{|f(x)| : x \in [-n,n]\}, \quad n \in \mathbb{N},$$
and an induced translation-invariant metric given by
$$d(f,g) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(f-g)}{1+p_n(f-g)}.$$
Define
$$f(x) = \max\{0, 1-|x|\}, \quad g(x) = 100 f(x-2), \quad h(x) = \frac{1}{2}\big(f(x)+g(x)\big),$$
and show that
$$d(f,0) = \frac{1}{2}, \quad d(g,0) = \frac{50}{101}, \quad d(h,0) = \frac{1}{6} + \frac{50}{102}.$$
Hence the ball $B(0,\frac{1}{2})$ is not convex.
Remark: One can show that $B(0,r)$ is not convex for any $0 < r < 1$.